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MOTION OF A VISCO-ELASTIC SPHERE IN A CENTRAL NEWTONIAN FORCE FIELD*

V. G. VIL'KE

Approximate equations are obtained, describing the evolution of translational-rotational motion of a visco-elastic sphere in a central Newtonian force field, its steady-state motions are found, and their stability is investigated. The motion described can serve as one model of tidal phenomena under the motion of a planet of the solar system /1/. It has been shown previously /2/ that in the presence of energy dissipation a deformable planet tends towards a steady-state rotation around the attracting center, under which the planet's center of mass describes a circle, while its orientation in the orbital axes in unchanged.

1. Equations of motion. Let $\mathbf{R}(t)$ be the radius-vector of the center C of a sphere relative to inertial axes $O\,\xi_1\xi_2\xi_3$ with origin at the attracting center and let $\boldsymbol{\xi}(\mathbf{r},t)$ be the vector from point C to some point of the deformed sphere ($\mathbf{r}\in\Omega$, where Ω is the domain occupied by the sphere in the undeformed state). At point C we introduce a moving coordinate system $Cx_1x_2x_3$ connected with the changeable sphere by the condition

$$\boldsymbol{\xi}(\mathbf{r},t)=O(t)(\mathbf{r}+\mathbf{u}(\mathbf{r},t)),\quad \int_\Omega \operatorname{rot}\mathbf{u}\,dx=0 \tag{1.1}$$

Here $O(t)\in SO(3)$ is an orthogonal matrix specifying the rotation of a three-dimensional space. The second relation in (1.1) signifies that in an integral sense the sphere under deformations does not rotate relative to the moving axes. This fact permits us to apply the linear theory of elasticity of small deformations when studying the sphere's motion relative to coordinate system $Cx_1x_2x_3$. Relations (1.1) uniquely define the matrix $O(t)$ and the displacements $\mathbf{u}$ with respect to the specified field $\boldsymbol{\xi}(\mathbf{r},t)$. From the vector equation

$$\int_\Omega \operatorname{rot}[O^{-1}\boldsymbol{\xi}(\mathbf{r},t)-\mathbf{r}]\,dx=0$$

we find three parameters (the Euler angles, for example) specifying matrix $O(t)$, and, further, $\mathbf{u}=O^{-1}\boldsymbol{\xi}(\mathbf{r},t)-\mathbf{r}$.

The absolute acceleration of a point on the sphere, in projection onto the axes of the moving coordinate system $Cx_1x_2x_3$, and its virtual displacements are given by the relations

$$\mathbf{w}(\mathbf{r},t)=O^{-1}\mathbf{R}^{\cdot\cdot}+\boldsymbol{\varepsilon}\times(\mathbf{r}+\mathbf{u})+\boldsymbol{\omega}\times[\boldsymbol{\omega}\times(\mathbf{r}+\mathbf{u})]+2\boldsymbol{\omega}\times\mathbf{u}^{\cdot}+\mathbf{u}^{\cdot\cdot} \tag{1.2}$$

$$O^{-1}\delta\boldsymbol{\xi}=O^{-1}\delta\mathbf{R}+\delta\boldsymbol{\alpha}\times(\mathbf{r}\times\mathbf{u})+\delta\mathbf{u},\quad \delta\boldsymbol{\alpha}\in R^3$$

Here $\boldsymbol{\varepsilon}$ and $\boldsymbol{\omega}$ are the angular acceleration and the angular velocity, respectively, of coordinate system $Cx_1x_2x_3$. The work of the active forces on the virtual displacements is prescribed in the form

$$\delta A=-\delta U-\delta E[\mathbf{u}]-(\nabla D[\mathbf{u}^{\cdot}],\delta\mathbf{u}),\qquad U=-\int_\Omega \frac{\mu\rho\,dx}{[(\mathbf{R}+O(\mathbf{r}+\mathbf{u}))^2]^{1/2}} \tag{1.3}$$

$$\delta U=(\nabla U[\mathbf{R}+O(\mathbf{r}+\mathbf{u})],\ \delta\mathbf{R}+O[\delta\boldsymbol{\alpha}\times(\mathbf{r}+\mathbf{u})]+O\delta\mathbf{u})\qquad \delta E[\mathbf{u}]=(\nabla E[\mathbf{u}],\delta\mathbf{u})=\int_\Omega\sum_{i,j=1}^{3}\frac{\partial e}{\partial u_{ij}}\delta u_{ij}\,dx,$$

$$(\nabla D[\mathbf{u}^{\cdot}],\delta\mathbf{u})=\int_\Omega\sum_{m,n,i,j=1}^{3}d_{mnij}e_{mn}^{\cdot}\delta e_{ij}\,dx,\qquad e=\sum_{m,n,i,j=1}^{3}e_{mnij}e_{mn}e_{ij},\qquad e_{ij}=\frac{1}{2}(u_{ij}+u_{ji}),\quad u_{ij}=\frac{\partial u_i}{\partial x_j}$$

In (1.3) we have used a model of the linear theory of visco-elasticity of small deformations because the relative displacements of the sphere's points under deformations $\mathbf{u}(\mathbf{r},t)$ are small. The coefficients a_{mnij} and d_{mnij} are constant and symmetric with respect to the first two and the last two indices, and the quadratic forms corresponding to them are positive definite in the variables e_{ij} and $e_{ij}^{\cdot}$, respectively. The sphere is assumed to be homogeneous and isotropic with constant density ρ. The constant μ characterizes the intensity of the gravitational field.

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Allowing for (1.2), (1.3) and relations (1.1), we represent the D'Alembert—Lagrange variational principle as

$$\int_\Omega [O^{-1}\mathbf{R}^{\cdot\cdot} + \varepsilon\times(\mathbf{r}+\mathbf{u}) + \omega\times(\omega\times(\mathbf{r}+\mathbf{u})) + 2\omega\times\mathbf{u}^{\cdot} + \mathbf{u}^{\cdot\cdot}][O^{-1}\delta\mathbf{R} + \delta\alpha\times(\mathbf{r}+\mathbf{u}) + \delta\mathbf{u}]\rho\, dx + \tag{1.4}$$
$$(O^{-1}\nabla U[\mathbf{R}+O(\mathbf{r}+\mathbf{U})], O^{-1}\delta\mathbf{R} + \delta\alpha\times(\mathbf{r}+\mathbf{u}) + \delta\mathbf{u}) + (\nabla \mathrm{E}[\mathbf{u}], \delta\mathbf{u}) + (\nabla D[\mathbf{u}^{\cdot}], \delta\mathbf{u}) +$$
$$\lambda\int_\Omega \mathrm{rot}\,\delta\mathbf{u}\, dx = 0, \quad \forall\delta\mathbf{u}\in V$$

where λ is the undetermined Lagrange multiplier, $V = (W_2^1(\Omega))^3$ is the system's configuration space, and the variations $\delta\mathbf{R}$, $\delta\alpha$, $\delta\mathbf{u}(\mathbf{r}, t)$ are independent. The system's equations of motion

$$MO^{-1}\mathbf{R}^{\cdot\cdot} + \int_\Omega [\varepsilon\times\mathbf{u} + \omega\times(\omega\times\mathbf{u}) + 2\omega\times\mathbf{u}^{\cdot} + \mathbf{u}^{\cdot\cdot}]\rho\, dx + \int_\Omega O^{-1}\nabla U[\mathbf{R}+O(\mathbf{r}+\mathbf{u})]\, dx = 0, \quad M = \int_\Omega \rho\, dx \tag{1.5}$$

$$\int_\Omega (\mathbf{r}+\mathbf{u})\times[O^{-1}\mathbf{R}^{\cdot\cdot} + \varepsilon\times(\mathbf{r}+\mathbf{u}) + \omega\times(\omega\times(\mathbf{r}+\mathbf{u})) + 2\omega\times\mathbf{u}^{\cdot} + \mathbf{u}^{\cdot\cdot}]\rho\, dx + \int_\Omega (\mathbf{r}+\mathbf{u})\times O^{-1}\nabla U[\mathbf{R}+O(\mathbf{r}+\mathbf{u})]\, dx = 0$$

$$\int_\Omega [O^{-1}\mathbf{R}^{\cdot\cdot} + \varepsilon\times(\mathbf{r}+\mathbf{u}) + \omega\times(\omega\times(\mathbf{r}+\mathbf{u})) + 2\omega\times\mathbf{u}^{\cdot} + \mathbf{u}^{\cdot\cdot}]\times\delta\mathbf{u}\rho\, dx + (\nabla \mathrm{E}[\mathbf{u}], \delta\mathbf{u}) + \int_\Omega O^{-1}\nabla U[\mathbf{R}+O(\mathbf{r}+\mathbf{u})]\times$$
$$\delta\mathbf{u}\, dx + (\nabla D[\mathbf{u}^{\cdot}], \delta\mathbf{u}) + \int_\Gamma (\lambda\times\mathbf{n})\,\delta\mathbf{u}\, d\sigma = 0, \quad \forall\delta\mathbf{u}\in V$$

follow from (1.4). In the third equation in (1.5), when transforming the last integral, we used the Ostrogradskii—Gauss formula(*) in the form

$$\int_\Omega \lambda\,\mathrm{rot}\,\delta\mathbf{u}\, dx = \int_\Gamma (\delta\mathbf{u}\times\lambda)\,\mathbf{n}\, d\sigma$$

where Γ is the sphere's surface and $\mathbf{n}$ is the normal to the sphere's surface. Under conditions (1.1) Eqs. (1.5) are the exact equations of motion of the deformable sphere in a Newtonian force field within the framework of the linear theory of visco-elasticity. They are valid for a deformable body of arbitrary form. When $\mathbf{u}\equiv 0$ the first two equations in (1.5) describe the translational-rotational motion of a rigid body in a Newtonian force field /4/. In the case of a hmogeneous sphere they can be integrated exactly: the sphere's center moves on a Keplerian orbit, while the sphere itself rotates with constant angular velocity around an invariable axis. With a great degree of accuracy a planet in the solar system has a similar motion. Let us study the effect of elasticity of the sphere on this motion.

Let the following inequalities hold:

$$\omega \ll \nu_1, \quad 2\pi/T \ll \nu_1, \quad \omega \ll \chi_1, \quad 2\pi/T \ll \chi_1, \quad |\mathbf{u}(\mathbf{r}, t)| \ll |\mathbf{r}|, \quad r_0 \ll |\mathbf{R}(t)| \tag{1.6}$$

Here ω, T, $\mathbf{R}(t)$ are the sphere's angular rotation velocity, the period, and the radius-vector of Keplerian motion, respectively, ν_1, χ_1 are the frequency and the damping decrement of the sphere's natural oscillations at the lowest harmonic, r_0 is the sphere's radius. Inequalities (1.6) permit us to neglect the inertial terms in the third equation in (1.5), to replace $\mathbf{r}+\mathbf{u}$ by $\mathbf{r}$, and to obtain an approximate equation in $\mathbf{u}(\mathbf{r}, t)$ describing the quasi-static process of deformation of the sphere

$$(\nabla \mathrm{E}[\mathbf{u}] + \nabla D[\mathbf{u}^{\cdot}], \delta\mathbf{u}) + \int_\Omega \{\rho\omega\times(\omega\times\mathbf{r}) + O^{-1}\nabla^2 U[\mathbf{R}]O\mathbf{r}\}\,\delta\mathbf{u}\, dx + \int_\Gamma (\lambda\times\mathbf{n})\,\delta\mathbf{u}\, d\sigma = 0, \quad \forall\delta\mathbf{u}\in V \tag{1.7}$$

Here the quantities ω, $O(t)$, $\mathbf{R}(t)$ correspond to the generating solution: $O(t)$ is the matrix of transfer to the axes connected with the sphere and rotating with angular velocity $\omega = \mathrm{const}$, $\mathbf{R}(t)$ is the radius-vector of the sphere's center of mass when moving on a Keplerian orbit. In what follows we shall take it that these quantities evolve. Only the linear terms relative to $\mathbf{r}$ in the expansion of the gradient of functional $U[\mathbf{R}+O\mathbf{r}]$ is retained in (1.7). We set $\delta\mathbf{u} = \delta\alpha\times\mathbf{r}$ and we note that all terms but the last in (1.7) vanish since the work of the elastic and the dissipative forces on infinitesimal rotations equals zero and the operators $\omega\times(\omega\times)$ and $O^{-1}\nabla^2 U[\mathbf{R}]O$ are symmetric. Then

$$\int_\Gamma (\lambda\times\mathbf{n})(\delta\alpha\times\mathbf{r})\, d\sigma = 4\pi r_0^3(\lambda, \delta\alpha) = 0, \quad \forall\delta\alpha\in R^3$$

and $\lambda = 0$. Equation (1.7) becomes linear, which permits us to seek its solution as a sum of particular solutions each of which is generated by an appropriate force field. The force field $\omega\times(\omega\times\mathbf{r})\rho$ is the field of centrifugal inertia forces when the coordinate system

*) Editor's Note: English equivalent is "Gauss divergence formula".

rotates with angular velocity ω. The field of the gradient of the gravitational forces is

$$O^{-1}\nabla^2 U[\mathbf{R}]O\mathbf{r} = -\frac{2\mu\rho\mathbf{r}}{R^3} - \frac{3\mu\rho}{R^3}[O^{-1}\mathbf{R}^\circ\times(O^{-1}\mathbf{R}^\circ\times\mathbf{r})], \qquad \mathbf{R}^\circ = \frac{\mathbf{R}}{R} \tag{1.8}$$

The first term in (1.8) corresponds to a spherically symmetric field and generates a spherically symmetric deformation of the sphere, while the second term specifies an axisymmetric field (the symmetry axis is determined by the moving vector $O^{-1}\mathbf{R}^\circ$) opposite to the field of centrifugal forces due to the rotation with angular velocity $(3\mu R^{-3})^{1/2}$.

We seek the solution of (1.7) as the sum $\mathbf{u} = \mathbf{u}_1(\mathbf{r}) + \mathbf{u}_2(\mathbf{r}, t) + \mathbf{u}_3(\mathbf{r}, t)$, where the $\mathbf{u}_i$ $(i = 1, 2, 3)$ satisfy the equations

$$(\nabla \mathrm{E}[\mathbf{u}_1], \delta\mathbf{u}) + \int_\Omega [\omega\times(\omega\times\mathbf{r})]\rho\delta\mathbf{u}\,dx = 0, \quad (\nabla\mathrm{E}[\mathbf{u}_2] + \nabla D[\mathbf{u}_2^{\cdot}], \delta\mathbf{u}) - \int_\Omega \frac{2\mu\rho}{R^3}\mathbf{r}\delta\mathbf{u}\,dx = 0, \quad \forall\delta\mathbf{u}\in V \tag{1.9}$$

$$(\nabla\mathrm{E}[\mathbf{u}_3] + \nabla D[\mathbf{u}_3^{\cdot}], \delta\mathbf{u}) - \int_\Omega \frac{3\mu\rho}{R^3}[O^{-1}\mathbf{R}^\circ(O^{-1}\mathbf{R}^\circ\times\mathbf{r})]\delta\mathbf{u}\,dx = 0$$

The first equation in (1.9) corresponds to the problem on the deformation of a uniformly rotating sphere and, in the case of a homogeneous isotropic sphere, is written in the Lamé form /5/

$$\Delta\mathbf{u}_1 + \frac{1}{1-2\nu}\operatorname{grad}\operatorname{div}\mathbf{u}_1 - \mathbf{e}_\omega\times(\mathbf{e}_\omega\times\mathbf{r})\omega^2 p_1 = 0, \qquad p_1 = \rho\frac{2(1+\nu)}{E} \tag{1.10}$$

where E and ν are the Young's modulus and the Poisson ratio, respectively, $\mathbf{e}_\omega$ is the unit vector along the sphere's rotation axis. The boundary conditions for the function $\mathbf{u}_1$ are that the stresses on the sphere's surface equal zero $(\sigma_n = 0)$. The solution of the sphere deformation problem in the absence of mass forces was obtained by Thompson with the use of spherical functions /5/. The special form of the mass forces in problem (1.10) enables us to obtain its solution as a sum of homogeneous spherical functions of third and first orders

$$u = q_1 x\rho\omega^2/E, \quad v = q_1 y\rho\omega^2/E, \quad w = q_2 z\rho\omega^2/E \tag{1.11}$$

$$q_1 = b_1(x^2+y^2) + b_2 z^2 + c_1, \quad q_2 = a_1(x^2+y^2) + a_2 z^2 + c_2$$

Here u, v, w are the components of vector $\mathbf{u}_1$ in coordinate system $Cxyz$, a_1, a_2, b_1, b_2, c_1, c_2 are constants yet to be determined. The coordinate system is connected with the sphere when its center of mass moves on a Keplerian orbit, while the sphere itself rotates around the center of mass (around the axis Cz). Substituting (1.11) into (1.10) and imposing the boundary condition $\sigma_n = 0$ on the sphere's surface, we obtain a system of six algebraic equations with constant coefficients in $a_1, a_2, b_1, b_2, c_1, c_2$

$$(6+8\beta)b_1 + 2\beta a_1 = -2(1+\nu), \quad \beta = 1/(1-2\nu), \qquad 2\beta b_2 + 3(1+\beta)a_2 = 0, \quad \gamma = \beta\nu \tag{1.12}$$

$$(3+4\gamma)b_1 + \gamma a_1 = a_1 + 2(1+\gamma)b_2 + 3\gamma a_2, \qquad (2+\gamma)a_1 + b_2 + 4\gamma b_1 = 3(1+\gamma)a_2 + 2\gamma b_2$$

$$(1+2\gamma)c_1 + \gamma c_2 = -r_0^2[(3+4\gamma)b_1 + \gamma a_1], \qquad 2\gamma c_1 + (1+\gamma)c_2 = -r_0^2[(2+\gamma)a_1 + b_2 + 4\gamma b_1]$$

It is quite cumbersome to solve (1.12) in its general form. For a qualitative estimate of the coefficients and to simplify the calculations we take $\nu = 0.24$, which corresponds to the Poisson ratio for iron, and we find

$$a_1 = 0.7669, \quad a_2 = 0.31117, \quad b_1 = -0.2539, \quad b_2 = -0.70947, \quad c_1 = 0.67453r_0^2, \quad c_2 = -0.91146r_0^2 \tag{1.13}$$

We find the solutions of the second and third equations in (1.9) under the assumption $D[\mathbf{u}] = \chi\mathrm{E}[\mathbf{u}]$ $(\chi \ll 1)$ since the functionals $\mathrm{E}[\mathbf{u}]$ and $D[\mathbf{u}]$ have like structure. This condition signifies that a proportional dependency exists between the Lamé elasticity coefficients and the viscosity coefficients. In view of the spherical symmetry of the force field the second equation in (1.9) has a spherically symmetric solution of form

$$\mathbf{u}_2(\mathbf{r}, t) = g(t)\varphi(\mathbf{r})\mathbf{r}, \quad g(t) = \frac{2\mu\rho}{R^3} + \chi\frac{d}{dt}\left(\frac{2\mu\rho}{R^3}\right) \tag{1.14}$$

We find the solution of the third equation in (1.9) in the form

$$\mathbf{u}_3(\mathbf{r}, t) = \sum_{n=0}^{\infty}(-1)^n\chi^n\frac{\partial^n\mathbf{u}_{30}(\mathbf{r}, t)}{\partial t^n} \tag{1.15}$$

where $\mathbf{u}_{30}(\mathbf{r}, t)$ is a solution of the equation

$$(\nabla\mathrm{E}[\mathbf{u}_{30}], \delta\mathbf{u}) - \frac{3\mu\rho}{R^3}\int_\Omega [O^{-1}\mathbf{R}^\circ\times(O^{-1}\mathbf{R}^{(0)}\times\mathbf{r})]\delta\mathbf{u}\,dx = 0, \quad \forall\delta\mathbf{u}\in V \tag{1.16}$$

The structure of (1.16) is analogous to that of (1.10). In the orbital coordinate system $Cx'y'z'$ (axis Cz' is directed along $O^{-1}\mathbf{R}^0$ and axis Cy' is orthogonal to the orbital plane) the solution of (1.16) can be written as

$$u_{30}' = q_1'x'\alpha(t), \quad v_{30}' = q_1'y'\alpha(t), \quad w_{30}' = q_2'z'\alpha(t), \quad \alpha(t) = -3\mu\rho/(ER^3) \tag{1.17}$$

Taking χ as a sufficiently small quantity and desiring to obtain qualitative results, we restrict ourselves in (1.15) to the first two terms and, as an approximate solution of the third equation in (1.9), we obtain the function

$$\mathbf{u}_3(\mathbf{r}, t) = A^{-1}(t)\,\mathbf{u}_{30}'(A(t)\mathbf{r}, t) - \chi\frac{d}{dt}[A^{-1}(t)\,\mathbf{u}_{30}'(A(t)\mathbf{r}, t)] \tag{1.18}$$

Here $A(t)$ is the orthogonal matrix of transition from the axes of $Cxyz$ to the orbital coordinate system $Cx'y'z'$. In the orbital coordinate system solution (1.18) becomes

$$\mathbf{u}_3'(\mathbf{r}', t) = (\alpha - \chi\alpha^{\cdot})[(B_1\mathbf{r}', \mathbf{r}')B_2 + (B_3\mathbf{r}', \mathbf{r}')B_4 + r_0^2B_5]\mathbf{r}' + \alpha\chi[(B_1\mathbf{r}', \mathbf{r}')(SB_2 - B_2S) + \tag{1.19}$$

$$(B_3\mathbf{r}', \mathbf{r}')(SB_4 - B_4S) + r_0^2(SB_5 - B_5S)]\mathbf{r}' - 2\alpha\chi[(B_1S\mathbf{r}', \mathbf{r}')B_2 + (B_3S\mathbf{r}', \mathbf{r}')B_4]\mathbf{r}'$$

$$B_1 = \operatorname{diag}\{b_1, b_1, b_2\}, \quad B_2 = \operatorname{diag}\{1, 1, 0\}, \quad B_3 = \operatorname{diag}\{a_1, a_1, a_2\}, \quad B_4 = \operatorname{diag}\{0, 0, 1\}$$

$$r_0^2B_5 = \operatorname{diag}\{c_1, c_1, c_2\}, \quad S = A^{\cdot}A^{-1} = -A(A^{-1})^{\cdot}$$

Here S is a skewsymmetric matrix characterizing the sphere's angular velocity $\mathbf{\Omega}^*$ relative to the orbital coordinate system $Cx'y'z'$. The mechanical sense of solution (1.18) is that the first term in it defines the tidal deformation along the axis connecting the center of mass with the attractive center and the second term characterizes the lag in the tidal deformation because of the viscous friction force.

Allowing for the property that the functions found are odd, $\mathbf{u}_i(\mathbf{r}, t) = -\mathbf{u}_i(-\mathbf{r}, t)$ $(i = 1, 2, 3)$, we compute the integral containing $\mathbf{u}(\mathbf{r}, t)$ in the first two of Eqs. (1.5) and we obtain approximate equations describing the translational-rotational motion of the deformed sphere

$$M\mathbf{R}^{\cdot\cdot} + \mathbf{R}^\circ\{M\mu R^{-2} + Q_2R^{-7} + 3\chi R^{-8}(\mathbf{R}^\circ, \mathbf{R}^{\cdot}) \times Q_2 + Q_1R^{-4}[\mathbf{L}^2 - 5(\mathbf{R}^\circ, \mathbf{L})^2]\} + 2Q_1R^{-4}(\mathbf{L}, \mathbf{R}^\circ)\mathbf{L} - \tag{1.20}$$

$$\chi Q_3R^{-7}[\mathbf{R}^\circ \times (\mathbf{L} - J_0\mathbf{\Omega}')] = 0, \quad \mathbf{L}^{\cdot} + \chi Q_3R^{-6}[\mathbf{L} - J_0\mathbf{\Omega}' - (\mathbf{L}, \mathbf{R}^\circ)\mathbf{R}^\circ] + 2Q_1R^{-3}(\mathbf{L}, \mathbf{R}^\circ)(\mathbf{L} \times \mathbf{R}^\circ) = 0$$

Here

$$Q_1 = \frac{3\mu\rho^2}{J_0^2E}\int_\Omega (q_1x^2 - q_2z^2)\,dx, \qquad Q_2 = \frac{18\mu^2\rho^2}{E}\int_\Omega (q_1x^2 - q_2z^2)\,dx$$

$$Q_3 = \frac{9\mu^2\rho^2}{J_0E}\int_\Omega \{[(B_1\mathbf{r}, \mathbf{r}) - (B_3\mathbf{r}, \mathbf{r}) + c_1 - c_2](x^2 + z^2) + 2(b_1 - b_2 + a_1 - a_2)x^2z^2\}\,dx$$

In Eqs. (1.20) $\mathbf{L}$ is the angular momentum of the sphere relative to the Koenig axes ($\mathbf{L} \approx J_0\omega$), J_0 is the moment of inertia of the undeformed sphere relative to the axis passing through the center of mass, Ω' is the sphere's angular rotation velocity relative to the orbital axes $Cx'y'z'$. When computing the integrals in (1.5) only the terms linear in $\mathbf{u}$ were left. The vector-valued quantities in (1.20) are taken in the inertial coordinate system.

2. Steady-state motions and their stability. Equations (1.20) enable us to find the evolution of the translational-rotational motion of a visco-elastic sphere in a Newtonian force field. They have the first integral, i.e., the law of conservation of angular momentum

$$\mathbf{R} \times M\mathbf{R}^{\cdot} + \mathbf{L} = \mathbf{G} \tag{2.1}$$

The sphere's steady-state motions are possible only when the forces determining energy dissipation vanish (the terms containing the multiplier χ in (1.20)) /1/

$$3R^{-8}(\mathbf{R}^\circ, \mathbf{R}^{\cdot})Q_2\mathbf{R}^\circ + Q_3R^{-7}[\mathbf{R}^\circ \times (\mathbf{L} - J_0\mathbf{\Omega}')] = 0, \qquad Q_3R^{-6}[\mathbf{L} - J_0\mathbf{\Omega}' - (\mathbf{L}, \mathbf{R}^\circ)\mathbf{R}^\circ] = 0 \tag{2.2}$$

From the first of Eqs. (2.2) follows $(\mathbf{R}^\circ, \mathbf{R}^{\cdot}) = 0$. We multiply the first of Eqs. (2.2) scalarly by $\mathbf{R}^{\cdot} = R(\mathbf{R}^\circ)^{\cdot}$ and the second by $J_0^{-1}\mathbf{L}$, we add the results, and, after manipulations, we obtain

$$(\mathbf{L} - J\mathbf{\Omega}')^2 + (\mathbf{L}, \mathbf{R}^\circ)^2 = 0$$

Consequently, $\mathbf{L} = J\mathbf{\Omega}'$ and vector $\mathbf{L}$ is orthogonal to $\mathbf{R}^\circ$. Using (2.1), we arrive at the following conclusion: in steady-state motion the sphere's center of mass moves along a circular orbit, while the deformed sphere is stationary in the orbital coordinate system.

The steady-state values of R and L are found from the equations

$$L^2 = \frac{1}{R^3}\left(M\mu + \frac{Q_2}{R^5}\right)\left(\frac{M}{J_0} - \frac{Q_1}{R^5}\right)^{-1}, \quad L = G\left(1 + \frac{MR^2}{J_0}\right)^{-1} \tag{2.3}$$

Numerical calculation shows that $Q_i > 0$ $(i = 1, 2, 3)$. Depending on the magnitude of G, Eq. (2.3) can have either two (when $G > G^*$) or one (when $G = G^*$) or no (when $G < G^*$) solutions

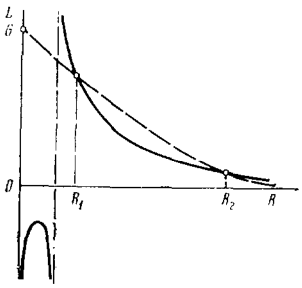

Fig.1

(the solid lines on the Fig.1 show the dependence of L on R according to the first equation in (2.3); the dashed line, according to the second equation).

Let us show that the steady-state motion is unstable for the smaller root R_1 and is stable for the larger root R_2. The equations in variations for the second of Eqs. (1.20) and for integral (2.1) have the form

$$\mathbf{l}^{\boldsymbol{\cdot}} + \chi Q_3 R^{-6}[\mathbf{l} - J_0\mathbf{k} - \varkappa(\mathbf{l}, \mathbf{R}^0)\mathbf{R}^0] + 2\varkappa Q_1 R^{-3}(\mathbf{L} \times \mathbf{R}^\circ) \times \tag{2.4}$$
$$(\mathbf{l}, \mathbf{R}^\circ) = 0, \quad 2MR\Omega'\xi + MR^2\mathbf{k} + \mathbf{l} = 0, \quad \varkappa = 1 + J_0 M^{-1} R^{-2}$$

Here $\mathbf{l}, \xi, \mathbf{k}$ are the variations of quantities $\mathbf{L}, \mathbf{R}, \Omega'$, respectively. The area constant G does not vary. Eliminating $\mathbf{k}$ from the first equation in (2.4) and projecting the resulting equation onto the axes of the orbital coordinate system, we obtain

$$l_x^{\boldsymbol{\cdot}} + a_{11}l_x + a_{13}l_z = 0, \quad l_y^{\boldsymbol{\cdot}} + a_{22}l_y + a_{24}\xi = 0 \tag{2.5}$$
$$l_z^{\boldsymbol{\cdot}} + a_{33}l_z + a_{31}l_x = 0, \quad a_{11} = a_{22} = \chi Q_3 R^{-6}(1 + J_0 M^{-1} R^{-2})$$
$$a_{13} = \Omega' + 2Q_1 R^{-3} L\varkappa, \; a_{33} = 0, \; a_{24} = 2\chi Q_3 J_0 \Omega' R^{-7}, \; a_{31} = -\Omega'$$

The quantities l_x, l_y, l_z are the projections of vector $\mathbf{l}$ onto the orbital axes $Cx'y'z'$ rotating with constant angular velocity Ω' around axis Cy'. The first and third equations in (2.5) are independent of the second. From them it follows that l_x and l_z tend to zero (a focus-type stable point).

When $l_x = l_z = 0$ and $l_y \neq 0$ the perturbed motion becomes planar since according to (2.1) the radius-vector $\mathbf{R}$ is then orthogonal to $\mathbf{G}$. In this case the equations of perturbed motion in projections onto the axes of the orbital coordinate system can be written as (the first equation in (1.20) and the projection of the second equation in (1.20) onto axis Cy')

$$M\xi^{\boldsymbol{\cdot\cdot}} + 3\chi Q_2 R^{-8}\xi^{\boldsymbol{\cdot}} + c\xi - 2MR\Omega'\eta + 2Q_1 R^{-4} L l_y = 0, \quad 2M\Omega'\xi^{\boldsymbol{\cdot}} + MR\eta^{\boldsymbol{\cdot}} + \chi Q_3 R^{-7} J_0 \eta - \chi Q_3 R^{-7} l_y = 0 \tag{2.6}$$
$$l_y^{\boldsymbol{\cdot}} + \chi Q_3 R^{-6} l_y - \chi Q_3 R^{-6} J_0 \eta = 0, \quad c = -2M\mu R^{-3} - M\Omega'^2 - 7Q_2 R^{-8} - 4Q_1 R^{-5} L^2$$

Here ξ and η are the variations of R and Ω', respectively $(\eta \mathbf{e}_y = \mathbf{k})$. The characteristic equation of system (2.6) is

$$D(a_0 D^3 + a_1 D^2 + a_2 D + a_3) = 0, \quad a_0 = M^2 R^2, \quad a_1 = \chi M R^{-6}[3Q_2 + Q_3(MR^2 + J_0)] \tag{2.7}$$
$$a_2 = MR^2(c + 4M\Omega'^2) + 3\chi^2 Q_2 Q_3 R^{-14}(MR^2 + J_0),$$
$$a_3 = \chi Q_3 R^{-6}[c(MR^2 + J_0) + 4M^2R^2\Omega'^2(1 - Q_1 J_0^2 R^{-5} M^{-1})]$$

The quantities R, Ω', c in (2.7) correspond to steady-state motion. Equation (2.7) has one zero root, which is a consequence of the law of conservation of angular momentum. The remaining roots have negative real parts for the steady-state motion with orbit radius R_2, which follows from the validity of the Hurwitz criterion inequalities $a_i > 0$ $(i = 0, 1, 2, 3)$ and $a_1 a_2 > a_0 a_3$. The approximate equalities $\mu R_2^{-3} \approx \Omega_2'^2$ and $c_2 \approx -3M\Omega_2'^2$ are valid for root R_2 and further

$$a_2 \approx M^2 R_2^2 \Omega_2'^2 + \chi^2 3Q_2 Q_3 R_2^{-14}(MR_2^2 + J_0) > 0$$
$$a_3 \approx \chi Q_3 R_2^{-4} M^2 \Omega_2'^2 (1 - 3J_0 M^{-1} R_2^{-2} - 4Q_1 J_0^2 M^{-1} R_2^{-5}) > 0$$
$$a_1 a_2 - a_0 a_3 > \chi M^2 R_2^{-4}[4Q_3 J_0 M \Omega_2'^2 + 3Q_2(c_2 + 4M\Omega_2'^2)] > 0$$

The coefficient $a_3 < 0$ for root R_1 since $R_1^5 \approx Q_1 J_0^2 M^{-1}$ and $a_3 \approx -\chi Q_3 R_1^{-6}(MR_1^2 + J_0)(2M\mu R_1^{-3} + 5M\Omega_1'^2 + 7Q_2 R_1^{-8}) < 0$. Thus, the steady-state motion with orbit radius R_2 is stable, while with orbit radius R_1 is unstable.

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